Saturation of a High-Gain Single-Pass FEL

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Abstract

We study a perturbation expansion for the solution of the nonlinear one-dimensional FEL equations. We show that in the case of a monochromatic wave, the radiated intensity satisfies a scaling relation that implies, for large distance z traveled along the undulator, a change in initial value of the radiation field corresponds to a translation in z (lethargy). Analytic continuation using Pade´ approximates yields accurate results for the radiation field early in saturation.

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1. Introduction

Free-Electron Laser (FEL) amplifiers in the exponential growth regime are accurately described by linear equations that are very well understood [1]. On the other hand, although there has been interesting work [2-7] on the theory of the saturation of the gain process, the description of the nonlinear phenomena involved is in a less advanced state. At present, most studies of saturation are based upon computer simulation [1]. In this paper, we use a perturbation expansion to treat the nonlinearity in the one-dimensional free-electron laser equations. For a monochromatic wave, the resulting Taylor series for the radiation field has a finite radius of convergence. We find that analytic continuation using Pade′ approximates [8] yields results in agreement with numerical integration of the 1-D FEL equations, well into saturation.

A more detailed exposition of the work reported in this paper as well as discussion of a simplified model for SASE statistics in saturation can be found in [9].

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2. Perturbation Expansion

The scaled equations [2] for the evolution of a one-dimensional electron distribution and a monochromatic radiation field are:

$$d\theta_j / dZ = p_j , \tag{1}$$

$$dp_j / dZ = -Ae^{i\theta_j} - A * e^{-i\theta_j} , \tag{2}$$

$$dA/dZ = \left\langle e^{-i\theta_j} \right\rangle . \tag{3}$$

$\theta_j = (k_s + k_w)z - \omega_s t_j(z)$ is the phase of the jth electron relative to the radiation and $p_j = (\gamma - \gamma_0)/\rho\gamma_0$ is its (scaled) energy deviation. We define : γ the relativistic parameter; $Z = 2\rho k_w z$ the scaled distance along the undulator axis; $2\pi/k_w$ the undulator period; $2\pi/k_s$ the radiation wavelength; and $t_j(z)$ the arrival time of the jth electron at position z. The radiated electric field has the form $E \exp[ik_s(z - ct)]$ and the scaled amplitude $A \equiv E / \sqrt{\rho n_0 \gamma_0 mc^2 / \varepsilon_0}$ (mks units), where ρ is the Pierce parameter and n_0 the electron density. The bracket $\langle \ \rangle$ indicates the average over the initial electron distribution.

We develop the solution of Eqs. (1-3) as a perturbation expansion in the small parameter ε, which we take to be the initial value of the radiation amplitude, $A(0) = \varepsilon$. Without loss of generality we consider $\varepsilon << 1$ to be real. Expanding in powers of ε, we write:

$$\theta(Z, \theta_0, p_0) = \theta_0 + p_0 Z + \varepsilon \theta_1(Z, \theta_0, p_0) + \varepsilon^2 \theta_2(Z, \theta_0, p_0) + \cdots , \tag{4}$$

$$A(Z) = \varepsilon A_1(Z) + \varepsilon^3 A_3(Z) + \varepsilon^5 A_5(Z) + \cdots . \tag{5}$$

The constraints: $\theta_n(0) = \theta_n'(0) = 0$ ($n \geq 1$), and $A_1(0) = 1$, $A_n(0) = 0$ ($n \geq 3$) assure that $\theta(0) = \theta_0$, $\theta'(0) = p_0$, and $A(0) = \varepsilon$. For an initially uniform, monoenergetic ($p_0 = 0$) electron beam, and a monochromatic electromagnetic wave, the system is periodic so we can restrict our attention to the interval $0 \leq \theta_0 \leq 2\pi$. $\left\langle e^{-im\theta_0} \right\rangle = \delta_{m,0}$, where $\delta_{m,0}$ is the Kronecker delta which equals unity for $m = 0$ and vanishes for all $m \neq 0$.

Eqs. (1-3) imply:

$$\theta'' = -Ae^{i\theta} - A * e^{-i\theta} , \tag{6}$$

$$A''' - iA = iA * \left\langle e^{-2i\theta} \right\rangle - \left\langle \theta'^2 e^{-i\theta} \right\rangle . \tag{7}$$

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The prime denotes derivative with respect to Z. We insert the expansions of Eqs. (4) and (5) into Eqs. (6) and (7), and equate terms having equal powers of ε. The first-order amplitude has the well-known solution [1], $A_1(Z) = \left(e^{sZ} + e^{-s*Z} + e^{-iZ}\right)/3$ where $s = \left(\sqrt{3}+i\right)/2$. There are three modes: growing; decaying and oscillating. For $Z \gg 1$, the exponentially growing mode dominates, $\varepsilon A_1(Z) \approx A_L(Z) \equiv (\varepsilon/3)\exp(sZ)$, and the perturbation coefficients θ_n and A_n have the form:

$$\varepsilon^n \theta_n(Z,\theta_0) = \sum_{k=0}^{n} b(n,n-2k) A_L^{n-k}(Z) A_L *^k (Z) e^{i(n-2k)\theta_0} \quad (n \geq 1) \tag{8}$$

and

$$\varepsilon^{2m+1} A_{2m+1}(Z) = a(m) A_L(Z) \left|A_L(Z)\right|^{2m} \quad (m \geq 0). \tag{9}$$

$b(n,n-2k)$ and $a(m)$ are complex constants independent of Z, determined recursively from Eqs. (6) and (7). We know that $a(0) = 1$ and find θ_1, θ_2 from Eq. (6) and then A_3 from Eq. (7). Next, θ_3, θ_4 are determined from Eq. (6). Once this is accomplished, A_5 is found from Eq. (7). In general, suppose we know $\theta_1, \theta_2, \cdots, \theta_{2m}$ and $A_1, A_3, \cdots, A_{2m+1}$, then θ_{2m+1} and θ_{2m+2} can be determined from Eq. (6), and then A_{2m+3} can be found from Eq. (7).

It is seen from Eqs. (5) and (9) that the radiation amplitude can be expressed in terms of the linear solution, $A_L(Z) = (\varepsilon/3)\exp(sZ)$, as

$$A(Z;\varepsilon) \cong A_L(Z) h(\left|A_L(Z)\right|^2), (Z \gg 1) \tag{10}$$

with

$$h(\xi) = \sum_{m=0}^{\infty} a(m)\xi^m, \tag{11}$$

Using Mathematica we have computed the coefficients a(1),…,a(12) of the power series in Eq. (11). In Table 1, columns 2 and 3, we present the magnitude and argument of the complex ratios, $a(n)/a(n-1)$. We see that after the first few values of n, the argument of this ratio remains close to 2.397 rad. The magnitude of the ratio also varies slowly. The variation is further reduced if we multiply by $n/(n-1/2)$. These results suggest that there exists an inverse square root branch point at $\xi_0 \cong \exp(-i2.397)/0.354$. This singularity limits the radius of convergence of the power series in Eq. (11). Therefore in order to use it to study saturation, we need to carry out an analytic continuation. A Taylor series can be analytically continued by the use of Padé

approximates [8]. One constructs a sequence of rational functions to approximate the unknown function such that when the rational functions are expanded, the coefficients match the original series expansion as well as possible.

For $Z \gg 1$, Eq. (10) implies that the radiation intensity has the form:

$$\left| A(\varepsilon, Z) \right|^2 \cong I(\xi) \equiv \xi \left| \sum_{m=0}^{\infty} a(m) \xi^m \right|^2 , \tag{12}$$

where the coefficients a(m) are complex and the scaling variable,

$$\xi \equiv \frac{1}{9} \varepsilon^2 e^{\sqrt{3} Z} , \tag{13}$$

is real. Eq. (12) shows that for large Z, the intensity does not depend on ε and Z independently, but only in the combination specified in Eq. (13). Therefore, a change in the initial value of the radiation field, ε, corresponds to a translation in Z. This is a mathematical expression of the intuitive idea that in a process with exponential growth the initial conditions are "forgotten." In FEL physics this property is sometimes referred to as "lethargy."

The singularity limits the radius of convergence of the power series in Eq. (11). Therefore in order to use it to study the saturation of the FEL, we need to carry out an analytic continuation. One approach to the analytic continuation of a Taylor series is the use of Pade$'$ approximates [8]. In this approach, one constructs a sequence of rational functions to approximate the unknown function. The rational functions are chosen such that when they are expanded, the coefficients match the original series expansion as well as possible. As an example [8], let us consider the function

$$f(x) = \left(\frac{1+2x}{1+x} \right)^{1/2} = 1 + \frac{1}{2} x - \frac{5}{8} x^2 + \frac{13}{16} x^3 - \frac{141}{128} x^4 + \cdots \tag{14}$$

Clearly, the Taylor series fails to converge for any value of x>1/2. The first Pade$'$ approximate is

$$\frac{1+(7/4)x}{1+(5/4)x} = 1 + \frac{1}{2} x - \frac{5}{8} x^2 + \frac{25}{32} x^3 - \frac{125}{128} x^4 + \cdots \tag{15}$$

This simple approximation has the value 1.4 at $x = \infty$ which should be compared to the exact value, $\sqrt{2}$. The next approximation is

$$\frac{1+(13/4)x+(41/16)x^2}{1+(11/4)x+(29/16)x^2} , \tag{16}$$

whose value is 1.4138 at $x = \infty$.

We expand the right-hand side of Eq. (12) in powers of ξ and analytically continue using Pade′ approximates. We denote by [M,N], the Pade′ approximate in which the numerator is a polynomial of degree M and the denominator is a polynomial of degree N. In Fig. 1, we plot the intensity $|A(Z)|^2$ versus Z for the [N,N] approximates, with N=1,…,6. It is seen that convergence out to about $Z = 10$ has been achieved for the [5,5] and [6,6] approximates. The [6,6] approximate agrees very accurately with the result of direct numerical integration of Eqs. (1-3) (dashed curve) out to Z=11. We have also used Eqs. (10) and (11) to calculate a power series expansion for the phase of the radiation field. We found that the [6,6] Pade′ approximation agrees very accurately with the numerical solution for the phase out to Z=10 [9].

3. Acknowledgement

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Table 1. Ratios of coefficients in expansion for $h(\xi)$ [Eq. (11)]

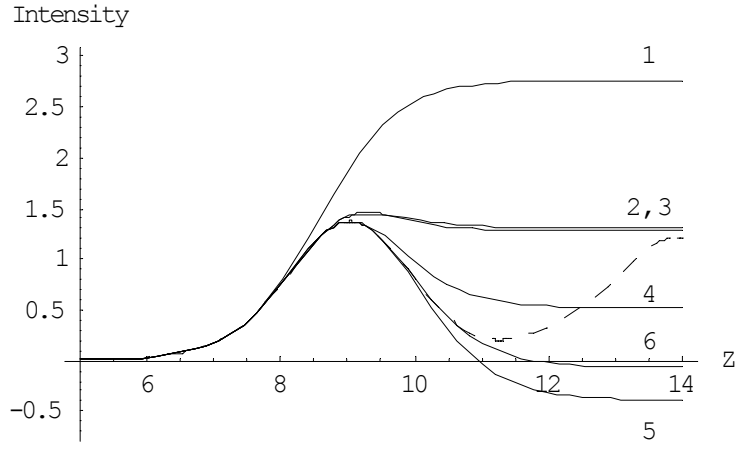| n	$\lvert a(n)/a(n-1) \rvert$	$Arg[a(n)/a(n-1)]$	$\lvert a(n)/a(n-1) \rvert \dfrac{n}{n-1/2}$
1	.216951	2.55393	.433903
2	.272966	2.43870	.363955
3	.298157	2.42034	.357788
4	.310309	2.40888	.354639
5	.318838	2.40122	.354264
6	.325133	2.39864	.354690
7	.329361	2.39838	.354696
8	.332254	2.39793	.354404
9	.334581	2.39709	.354262
10	.336581	2.39662	.354296
11	.338190	2.39659	.354294
12	.339444	2.39654	.354203

Fig. 1. The dimensionless intensity $\left|A(Z)\right|^2$ as derived from the [N,N] (N=1,…,6) Pade′ approximates (for $\varepsilon = .003$) versus dimensionless distance Z travelled along the undulator. The dashed curve shows the result of a numerical integration of Eqs. (1-3).